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COMMON FIXED POINTS FOR A COUNTABLE FAMILY OF NON-SELF MULTI-VALUED MAPPINGS ON METRICALLY CONVEX SPACES

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Abstract. In this paper, we will consider some existence theorems of common fixed points for a countable family of non-self multi-valued mappings defined on a closed subset of a complete metrically convex space, and give more generalized common fixed point theorems for a countable family of single-valued mappings. The main results in this paper generalize and improve many common fixed point theorems for single valued or multi-valued mappings with contractive type conditions.

1. Introduction

There are many fixed point theorems for a single-valued self map of a closed subset of a Banach space. However, in many applications, the mapping under considerations is a not self-mapping of a closed subset. Assad ([2]) gave a sufficient condition for such single valued mapping to obtain a fixed point by proving a fixed point theorem for Kannan mappings on a Banach space and putting certain boundary conditions on mapping. Similar results for multi-valued mappings were obtained by Assad ([1]) and Assad and Kirk ([3]). On the other hand, many authors discussed common fixed point problems ([7-8, 12, 14]) for finite single or multi-valued mappings on a complete 2-metric convex space or a complete cone metric space. And some authors also discussed common fixed point problems ([4-6, 9-11, 13]) for a countable family of

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self-single-valued mappings or non-self-single-valued mappings with certain boundary conditions on a metric space or a metrically convex space. These results improve and generalize many previous works.

In this paper, we will discuss the existence problems of common fixed points for a countable family of non-self multi-valued mappings defined on a closed subset of a complete metrically convex space, and obtain some interesting results. The main results in this paper generalize and improve many common fixed point theorems for single valued or multi-valued mappings with contractive type conditions.

Through this paper, (X, d) (or X) denotes a complete metric space. Let $bc(X)$ and $k(X)$ denote the families of all bounded closed subsets and compact subsets of X, respectively. Let H denote the Hausdorff metric on $bc(X)$, that is, for each $A, B \in bc(X)$,

$$H(A,B) = \max\Big\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\Big\},$$

where $d(x, A) = \inf_{y\in A}\{d(x, y)\}$.

Definition 1.1. [4-6] A metric space (X, d) is said to be *metrically convex*, if for any $x, y \in X$ with $x \neq y$, there exists $z \in X$ such that $z \neq x$, $z \neq y$ and $d(x, z) + d(z, y) = d(x, y)$.

Lemma 1.2. [3, 6] *If K is a nonempty closed subset of a complete metrically convex space (X, d), then for any $x \in K$ and $y \notin K$, there exists $z \in \partial K$ such that $d(x, z) + d(z, y) = d(x, y)$.*

The following Lemma can be found in [13].

Lemma 1.3. *If X is a complete metric space and $A, B \in bc(X)$, then*

(i) *For any $\varepsilon > 0$ and any $a \in A$, there exists $b \in B$ such that $d(a, b) \le H(A, B) + \varepsilon$;*
(ii) *For any $\beta > 1$ and any $a \in A$, there exists $b \in B$ such that $d(a, b) \le \beta H(A, B)$;*
(iii) *If $A, B \in k(X)$, then for any $a \in A$, there exists $b \in B$ such that $d(a, b) \le H(A, B)$.*

2. Main Theorems

We will discuss existence problems of common fixed points for a countable family of non-self multi-valued mappings defined on a nonempty closed subset of a complete metrically convex space, and give its generalized results.

THEOREM 2.1. *Let K be a nonempty closed subset of a complete metrically convex space (X,d), $\{T_i : K \to bc(X)\}_{i\in\mathbb{N}}$ a countable family of non-self set-valued mappings with nonempty values such that for any $i,j \in \mathbb{N}$ with $i \neq j$, any $x,y \in K$ and any $u \in T_i x$, there exists $v \in T_j y$ satisfying*

$$d(u,v) \le \lambda u_{i,j}(x,y), \tag{2.1}$$

where
$u_{i,j}(x,y) \in \left\{ d(x,y), d(x,T_ix), d(y,T_jy), \frac{d(x,T_ix)+d(y,T_jy)}{2}, \frac{d(x,T_jy)+d(y,T_ix)}{2} \right\}$,
and $\lambda \in (0,\frac{1}{2})$ is a constant number. Furthermore, if $T_i(x) \subset K$ for all $i \in \mathbb{N}$ and $x \in \partial K$, then $\{T_i\}_{i\in\mathbb{N}}$ has a common fixed point in K. If the condition (2.1) holds for all $u \in T_ix$ and $v \in T_jy$, then $\{T_i\}_{i\in\mathbb{N}}$ has a unique common fixed point in K.

Proof. Take $x_0 \in K$. We will construct two sequences $\{x_n\}$ and $\{x'_n\}$ in the following way. Take an element $x'_1 \in T_1x_0$. If $x'_1 \in K$, then put $x_1 = x'_1$; if $x'_1 \notin K$, then by Lemma 1.2 there exists $x_1 \in \partial K$ such that

$$d(x_0,x_1) + d(x_1,x'_1) = d(x_0,x'_1).$$

For $x'_1 \in T_1x_0$, there exists $x'_2 \in T_2x_1$ satisfying the condition (2.1). If $x'_2 \in K$, then put $x_2 = x'_2$; if $x'_2 \notin K$, then by Lemma 1.2 there exists $x_2 \in \partial K$ such that

$$d(x_1,x_2) + d(x_2,x'_2) = d(x_1,x'_2).$$

Continuing this way, we will obtain $\{x_n\}$ and $\{x'_n\}$ as follows:

(i) let $x'_1 \in T_1x_0$ and for each $x'_n \in T_nx_{n-1}$, there exists $x'_{n+1} \in T_{n+1}x_n$ satisfying the condition (2.1) for all $n > 1$;

(ii) if $x'_n \in K$, then put $x_n = x'_n$;

(iii) if $x'_n \notin K$, then by Lemma 1.2 there exists $x_n \in \partial K$ such that $d(x_{n-1},x_n) + d(x_n,x'_n) = d(x_{n-1},x'_n)$.

Let $P = \{x_i \in \{x_n\} : x_i = x'_i\}$ and $Q = \{x_i \in \{x_n\} : x_i \neq x'_i\}$. If there exists $n \in \mathbb{N}$ such that $x_n \in Q$, then x_{n-1} and $x_{n+1} \in P$. If fact, if $x_{n-1} \in Q$, then $x_{n-1} \neq x'_{n-1}$. Hence $x_{n-1} \in \partial K$ and $x'_{n-1} \notin K$. So by the given boundary condition, we have that $x'_n \in T_nx_{n-1} \subset K$. On the other hand, since $x_n \in Q$, hence $x_n \in \partial K$ and $x'_n \notin K$ which is a contradiction. Similarly, we can prove that $x_{n+1} \in P$.

If $x'_1 = x_0$, then $x'_1 = x_0$ is a common fixed point of $\{T_i\}_{i\in\mathbb{N}}$, that is $x'_1 \in T_jx'_1$, for all $j \in \mathbb{N}$. In fact, for $x'_1 \in T_1x_0$, there exists $v \in T_jx_0$

$(j \neq 1)$ satisfying (2.1), i.e., $d(x_1', v) \leq \lambda u_{1,j}(x_0, x_0)$,

$$\begin{aligned}\text{where,}\ \ u_{1,j}(x_0,x_0)\ \in &\Big\{d(x_0,x_0), d(x_0,T_1x_0), d(x_0,T_jx_0),\\ &\frac{d(x_0,T_1x_0)+d(x_0,T_jx_0)}{2}, \frac{d(x_0,T_jx_0)+d(x_0,T_1x_0)}{2}\Big\}\\ =&\Big\{0, d(x_0,T_jx_0), \frac{d(x_0,T_jx_0)}{2}\Big\}.\end{aligned}$$

If $u_{1,j}(x_0, x_0) = 0$, then $d(x_0, v) \leq 0$ and so $v = x_0$.

If $u_{1,j}(x_0, x_0) = d(x_0, T_jx_0)$, then $d(x_0, v) \leq \lambda d(x_0, T_jx_0)$, and hence

$$d(x_0, v) \leq \lambda d(x_0, v).$$

Since $0 < \lambda < 1$, we have $d(x_0, v) = 0$, and therefore $v = x_0$.

If $u_{1,j}(x_0, x_0) = \frac{d(x_0,T_jx_0)}{2}$, then $d(x_0, v) \leq \frac{\lambda}{2} d(x_0, T_jx_0)$. Consequently we have $d(x_0, v) \leq \frac{\lambda}{2} d(x_0, v)$. Since $0 < \lambda < 1$, $d(x_0, v) = 0$, and so $v = x_0$.

In any case, we obtain that $v = x_0$, and so $x_0 \in T_jx_0$ for all $j \in \mathbb{N}$. This means that x_0 is a common fixed point of $\{T_i\}_{i\in\mathbb{N}}$. By the above fact, we can suppose that

$$x_n' \in T_nx_{n-1} \text{ and } x_n' \neq x_{n-1}, \quad n = 2, 3, \cdots$$

By the properties of P and Q, we can estimate $d(x_n, x_{n+1})$ into three cases:

Case I. Suppose $x_n, x_{n+1} \in P$. Then we have

$$x_n = x_n' \in T_nx_{n-1} \text{ and } x_{n+1} = x_{n+1}' \in T_{n+1}x_n.$$

Hence we get by (2.1) that

$$d(x_n, x_{n+1}) = d(x_n', x_{n+1}') \leq \lambda u_{n,n+1}(x_{n-1}, x_n),$$

$$\begin{aligned}\text{where,}\ \ &u_{n,n+1}(x_{n-1},x_n) \in \Big\{d(x_{n-1},x_n), d(x_{n-1},T_nx_{n-1}), d(x_n,T_{n+1}x_n),\\ &\frac{d(x_{n-1},T_nx_{n-1})+d(x_n,T_{n+1}x_n)}{2}, \frac{d(x_{n-1},T_{n+1}x_n)+d(x_n,T_nx_{n-1})}{2}\Big\}.\end{aligned}$$

If $u_{n,n+1}(x_{n-1}, x_n) = d(x_{n-1}, x_n)$, then

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n).$$

If $u_{n,n+1}(x_{n-1}, x_n) = d(x_{n-1}, T_nx_{n-1})$, then

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, T_nx_{n-1}).$$

Since $x_n \in T_nx_{n-1}$, we have

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n).$$

If $u_{n,n+1}(x_{n-1}, x_n) = d(x_n, T_{n+1}x_n)$, then

$$d(x_n, x_{n+1}) \leq \lambda d(x_n, T_{n+1}x_n).$$

Since $x_{n+1} \in T_{n+1}x_n$, we obtain

$$d(x_n, x_{n+1}) \leq \lambda d(x_n, x_{n+1}),$$

and therefore $d(x_n, x_{n+1}) = 0$ since $\lambda < 1$, so we have

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n).$$

If $u_{n,n+1}(x_{n-1}, x_n) = \frac{d(x_{n-1},T_n x_{n-1})+d(x_n,T_{n+1}x_n)}{2}$, then we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{\lambda}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &\leq \frac{\lambda}{2} d(x_{n-1}, x_n) + \frac{1}{2} d(x_n, x_{n+1}). \end{aligned}$$

Consequently we obtain that

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n).$$

If $u_{n,n+1}(x_{n-1}, x_n) = \frac{d(x_{n-1},T_{n+1}x_n)+d(x_n,T_n x_{n-1})}{2}$, then we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{\lambda}{2} d(x_{n-1}, x_{n+1}) \leq \frac{\lambda}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &\leq \frac{\lambda}{2} d(x_{n-1}, x_n) + \frac{1}{2} d(x_n, x_{n+1}) \end{aligned}$$

Hence we obtain that

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n).$$

Consequently we have

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n) \text{ for all } n \in \mathbb{N}.$$

Case II. Suppose $x_n \in P$ and $x_{n+1} \in Q$. Then we have

$$x_n = x'_n \in T_n x_{n-1} \text{ and } x_{n+1} \neq x'_{n+1} \in T_{n+1}x_n.$$

Hence by (2.1), we get

$$d(x_n, x'_{n+1}) = d(x'_n, x'_{n+1}) \leq \lambda u_{n,n+1}(x_{n-1}, x_n),$$

where, $u_{n,n+1}(x_{n-1}, x_n) \in \Big\{ d(x_{n-1}, x_n), d(x_{n-1}, T_n x_{n-1}), d(x_n, T_{n+1}x_n),$

$$\frac{d(x_{n-1}, T_n x_{n-1}) + d(x_n, T_{n+1}x_n)}{2}, \frac{d(x_{n-1}, T_{n+1}x_n) + d(x_n, T_n x_{n-1})}{2} \Big\}.$$

If $u_{n,n+1}(x_{n-1}, x_n) = d(x_{n-1}, x_n)$, then

$$d(x_n, x'_{n+1}) \leq \lambda d(x_{n-1}, x_n).$$

Since $d(x_n, x_{n+1}) + d(x_{n+1}, x'_{n+1}) = d(x_n, x'_{n+1})$, we have

$$d(x_n, x_{n+1}) \leq d(x_n, x'_{n+1}) \leq \lambda d(x_{n-1}, x_n).$$

If $u_{n,n+1}(x_{n-1}, x_n) = d(x_{n-1}, T_n x_{n-1})$, then

$$d(x_n, x'_{n+1}) \leq \lambda d(x_{n-1}, T_n x_{n-1}) \leq \lambda d(x_{n-1}, x_n).$$

Since $d(x_n, x_{n+1}) + d(x_{n+1}, x'_{n+1}) = d(x_n, x'_{n+1})$, we have

$$d(x_n, x_{n+1}) \leq d(x_n, x'_{n+1}) \leq \lambda d(x_{n-1}, x_n).$$

If $u_{n,n+1}(x_{n-1}, x_n) = d(x_n, T_{n+1} x_n)$, then

$$d(x_n, x'_{n+1}) \leq \lambda d(x_n, T_{n+1} x_n) \leq \lambda d(x_n, x'_{n+1}).$$

Since $\lambda < 1$, we have $d(x_n, x'_{n+1}) = 0$ and so $d(x_n, x_{n+1}) = 0$. Hence we have

$$d(x_n, x_{n+1}) \leq d(x_n, x'_{n+1}) \leq \lambda d(x_{n-1}, x_n).$$

If $u_{n,n+1}(x_{n-1}, x_n) = \frac{d(x_{n-1}, T_n x_{n-1}) + d(x_n, T_{n+1} x_n)}{2}$, then

$$\begin{aligned} d(x_n, x'_{n+1}) &\leq \frac{\lambda}{2}[d(x_{n-1}, T_n x_{n-1}) + d(x_n, T_{n+1} x_n)] \\ &\leq \frac{\lambda}{2}[d(x_{n-1}, x_n) + d(x_n, x'_{n+1})] \\ &\leq \frac{\lambda}{2} d(x_{n-1}, x_n) + \frac{1}{2} d(x_n, x'_{n+1}). \end{aligned}$$

So we have

$$d(x_n, x_{n+1}) \leq d(x_n, x'_{n+1}) \leq \lambda d(x_{n-1}, x_n).$$

If $u_{n,n+1}(x_{n-1}, x_n) = \frac{d(x_{n-1}, T_{n+1} x_n) + d(x_n, T_n x_{n-1})}{2}$, then

$$\begin{aligned} d(x_n, x'_{n+1}) &\leq \frac{\lambda}{2}[d(x_{n-1}, T_{n+1} x_n) + d(x_n, T_n x_{n-1})] \\ &\leq \frac{\lambda}{2} d(x_{n-1}, x'_{n+1}) \leq \frac{\lambda}{2}[d(x_{n-1}, x_n) + d(x_n, x'_{n+1})] \\ &\leq \frac{\lambda}{2} d(x_{n-1}, x_n) + \frac{1}{2} d(x_n, x'_{n+1}). \end{aligned}$$

So we have

$$d(x_n, x_{n+1}) \leq d(x_n, x'_{n+1}) \leq \lambda d(x_{n-1}, x_n).$$

Consequently, we have

$$d(x_n, x_{n+1}) \leq d(x_n, x'_{n+1}) \leq \lambda d(x_{n-1}, x_n) \quad \text{for all } n \in \mathbb{N}.$$

Case III. Suppose $x_n \in Q$ and $x_{n+1} \in P$. Then we have $x_{n-1} \in P$ by the property of P and Q, and so $x_n \neq x'_n \in T_n x_{n-1}$ and $x_{n+1} = x'_{n+1} \in T_{n+1} x_n$. By (2.1), we get

$$d(x'_n, x_{n+1}) = d(x'_n, x'_{n+1}) \leq \lambda u_{n,n+1}(x_{n-1}, x_n),$$

where, $u_{n,n+1}(x_{n-1}, x_n) \in \Big\{ d(x_{n-1}, x_n), d(x_{n-1}, T_n x_{n-1}), d(x_n, T_{n+1} x_n),$

$$\frac{d(x_{n-1}, T_n x_{n-1}) + d(x_n, T_{n+1} x_n)}{2}, \frac{d(x_{n-1}, T_{n+1} x_n) + d(x_n, T_n x_{n-1})}{2} \Big\}.$$

Here we will give two properties:

(a) $d(x_{n-1}, x_n) + d(x_n, x'_n) = d(x_{n-1}, x'_n)$ since $x_n \in Q$.

(b) since
$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, x'_n) + d(x'_n, x_{n+1}) \\ &\leq d(x_{n-1}, x_n) + d(x_n, x'_n) + d(x'_n, x_{n+1}) \\ &= d(x_{n-1}, x'_n) + d(x'_n, x_{n+1}). \end{aligned}$$

So we have

$$d(x_n, x_{n+1}) - d(x_{n-1}, x'_n) \leq d(x'_n, x_{n+1}).$$

If $u_{n,n+1}(x_{n-1}, x_n) = d(x_{n-1}, x_n)$, then

$$d(x'_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n).$$

Hence by (b), we have

$$d(x_n, x_{n+1}) - d(x_{n-1}, x'_n) \leq \lambda d(x_{n-1}, x_n).$$

So by (a), we get

$$d(x_n, x_{n+1}) \leq (1 + \lambda) d(x_{n-1}, x'_n).$$

By Case II, we have

$$d(x_n, x_{n+1}) \leq \lambda(1 + \lambda) d(x_{n-2}, x_{n-1}).$$

If $u_{n,n+1}(x_{n-1}, x_n) = d(x_{n-1}, T_n x_{n-1})$, then

$$d(x'_n, x_{n+1}) \leq \lambda d(x_{n-1}, T_n x_{n-1}) \leq \lambda d(x_{n-1}, x'_n).$$

Hence by (b) and Case II, we obtain

$$d(x_n, x_{n+1}) \leq \lambda(1 + \lambda) d(x_{n-2}, x_{n-1}).$$

If $u_{n,n+1}(x_{n-1}, x_n) = d(x_n, T_{n+1} x_n)$, then

$$d(x'_n, x_{n+1}) \leq \lambda d(x_n, T_{n+1} x_n) \leq \lambda d(x_n, x_{n+1}).$$

By (b), we get

$$(1 - \lambda) d(x_n, x_{n+1}) \leq d(x_{n-1}, x'_n).$$

So by Case II again, we obtain

$$d(x_n, x_{n+1}) \leq \frac{\lambda}{(1-\lambda)} d(x_{n-2}, x_{n-1}).$$

If $u_{n,n+1}(x_{n-1}, x_n) = \frac{d(x_{n-1}, T_n x_{n-1}) + d(x_n, T_{n+1} x_n)}{2}$, then

$$\begin{aligned} d(x'_n, x_{n+1}) &\leq \frac{\lambda}{2}[d(x_{n-1}, T_n x_{n-1}) + d(x_n, T_{n+1} x_n)] \\ &\leq \frac{\lambda}{2}[d(x_{n-1}, x'_n) + d(x_n, x_{n+1})]. \end{aligned}$$

By (b), we obtain

$$d(x_n, x_{n+1}) - d(x_{n-1}, x'_n) \leq \frac{\lambda}{2}[d(x_{n-1}, x'_n) + d(x_n, x_{n+1})].$$

Hence

$$d(x_n, x_{n+1}) \leq \frac{2+\lambda}{2-\lambda} d(x_{n-1}, x'_n),$$

and so by Case II again, we obtain that

$$d(x_n, x_{n+1}) \leq \frac{(2+\lambda)\lambda}{2-\lambda} d(x_{n-2}, x_{n-1}).$$

If $u_{n,n+1}(x_{n-1}, x_n) = \frac{d(x_{n-1}, T_{n+1} x_n) + d(x_n, T_n x_{n-1})}{2}$, then

$$\begin{aligned} d(x'_n, x_{n+1}) &\leq \frac{\lambda}{2}[d(x_{n-1}, T_{n+1} x_n) + d(x_n, T_n x_{n-1})] \\ &\leq \frac{\lambda}{2}[d(x_{n-1}, x_{n+1}) + d(x_n, x'_n)] \\ &\leq \frac{\lambda}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_n, x'_n)]. \end{aligned}$$

Hence by (a) and (b), we have that

$$d(x_n, x_{n+1}) - d(x_{n-1}, x'_n) \leq \frac{\lambda}{2}[d(x_{n-1}, x'_n) + d(x_n, x_{n+1})],$$

and so

$$d(x_n, x_{n+1}) \leq \frac{2+\lambda}{2-\lambda} d(x_{n-1}, x'_n).$$

By Case II, we obtain

$$d(x_n, x_{n+1}) \leq \frac{(2+\lambda)\lambda}{2-\lambda} d(x_{n-2}, x_{n-1}).$$

In any cases, we have

$$d(x_n, x_{n+1}) \leq \max\Big\{\lambda(1+\lambda), \frac{\lambda}{1-\lambda}, \frac{(2+\lambda)\lambda}{2-\lambda}\Big\} d(x_{n-2}, x_{n-1}), \quad \forall\, n \in \mathbb{N}, n \geq 2.$$

So from Case I, Case II and Case III, we have

$$d(x_n, x_{n+1}) \leq \max\Big\{\lambda, \lambda(1+\lambda), \frac{\lambda}{1-\lambda}, \frac{(2+\lambda)\lambda}{2-\lambda}\Big\} \times \max\{d(x_{n-1}, x_n), d(x_{n-2}, x_{n-1})\}, \quad \forall\, n \in \mathbb{N}, n \geq 2.$$

It is easy to check that

$$\max\Big\{\lambda, \lambda(1+\lambda), \frac{\lambda}{1-\lambda}, \frac{(2+\lambda)\lambda}{2-\lambda}\Big\} = \frac{\lambda}{1-\lambda},$$

and

$$d(x_n, x_{n+1}) \leq h \max\Big\{d(x_{n-1}, x_n), d(x_{n-2}, x_{n-1})\Big\}$$

for all $n \in \mathbb{N}, n \geq 2$, where $h = \frac{\lambda}{1-\lambda}$. Clearly h is an increasing function on $\lambda \in (0,1)$, and $h < 1$ if and only if $\lambda \in (0, \frac{1}{2})$. Hence we know that $0 < h < 1$ and we have

$$d(x_n, x_{n+1}) \leq h^{\frac{n}{2}-1} \max\{d(x_2, x_1), d(x_1, x_0)\},$$

for all $n \in \mathbb{N}$ and $n \geq 2$. Let $\delta = h^{-1} \max\{d(x_2, x_1), d(x_1, x_0)\}$. Then for $m > n \geq N \geq 2$,

$$d(x_n, x_m) \leq \Sigma_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \Sigma_{i=N}^{\infty} d(x_i, x_{i+1}) \leq \Sigma_{i=N}^{\infty} \Big(h^{\frac{1}{2}}\Big)^i \delta.$$

Hence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ has a limit x^*. But K is closed and $x_n \in K$ for all $n \in \mathbb{N}$, and hence $x^* \in K$. By the properties of P and Q, we can see that there exists an infinite subsequence $\{x_{n_k+1}\}$ of $\{x_n\}$ such that $x_{n_k+1} \in P$, and so $x_{n_k+1} = x'_{n_k+1} \in T_{n_k+1} x_{n_k}$.

Now we prove that x^* is a common fixed point of $\{T_i\}_{i\in\mathbb{N}}$. In fact, for any fixed $i \in \mathbb{N}$, we can take k large enough such that $n_k > i$ and $x_{n_k+1} \in P$. For $x_{n_k+1} = x'_{n_k+1} \in T_{n_k+1} x_{n_k}$, there exists $v \in T_i x^*$ satisfying (2.1), i.e.,

$$d(x_{n_k+1}, v) = d(x'_{n_k+1}, v) \leq \lambda u_{n_k+1,i}(x_{n_k}, x^*),$$

where, $u_{n_k+1,i}(x_{n_k}, x^*) \in \Big\{d(x_{n_k}, x^*), d(x_{n_k}, T_{n_k+1} x_{n_k}), d(x^*, T_i x^*),$

$$\frac{d(x_{n_k}, T_{n_k+1} x_{n_k}) + d(x^*, T_i x^*)}{2}, \frac{d(x_{n_k}, T_i x^*) + d(x^*, T_{n_k+1} x_{n_k})}{2}\Big\}.$$

If $u_{n_k+1,i}(x_{n_k}, x^*) = d(x_{n_k}, x^*)$, then

$$d(x_{n_k+1}, v) = d(x'_{n_k+1}, v) \leq \lambda d(x_{n_k}, x^*).$$

Let $k \to \infty$. Then $d(x^*, v) \leq \lambda d(x^*, x^*) = 0$, and so $d(x^*, v) = 0$, that is, $v = x^*$.

If $u_{n_k+1,i}(x_{n_k}, x^*) = d(x_{n_k}, T_{n_k+1}x_{n_k})$, then

$$d(x_{n_k+1}, v) = d(x'_{n_k+1}, v) \leq \lambda d(x_{n_k}, T_{n_k+1}x_{n_k}) \leq \lambda d(x_{n_k}, x_{n_k+1}).$$

Let $k \to \infty$. Then $d(x^*, v) \leq \lambda d(x^*, x^*) = 0$, and so $d(x^*, v) = 0$, that is, $v = x^*$.

If $u_{n_k+1,i}(x_{n_k}, x^*) = d(x^*, T_i x^*)$, then

$$d(x_{n_k+1}, v) = d(x'_{n_k+1}, v) \leq \lambda d(x^*, T_i x^*) \leq d(x^*, v).$$

Let $k \to \infty$. Then $d(x^*, v) \leq \lambda d(x^*, v)$, and so $d(x^*, v) = 0$ since $\lambda < 1$, so $v = x^*$.

If $u_{n_k+1,i}(x_{n_k}, x^*) = \frac{d(x_{n_k}, T_{n_k+1}x_{n_k}) + d(x^*, T_i x^*)}{2}$, then

$$\begin{aligned} d(x_{n_k+1}, v) &\leq \frac{\lambda}{2}[d(x_{n_k}, T_{n_k+1}x_{n_k}) + d(x^*, T_i x^*)] \\ &\leq \frac{\lambda}{2}[d(x_{n_k}, x_{n_k+1}) + d(x^*, v)]. \end{aligned}$$

Let $k \to \infty$. Then $d(x^*, v) \leq \frac{\lambda}{2} d(x^*, v)$, and so $d(x^*, v) = 0$ since $\lambda < 1$, so $v = x^*$.

If $u_{n_k+1,i}(x_{n_k}, x^*) = \frac{d(x_{n_k}, T_i x^*) + d(x^*, T_{n_k+1}x_{n_k})}{2}$, then

$$\begin{aligned} d(x_{n_k+1}, v) &\leq \frac{\lambda}{2}[d(x_{n_k}, T_i x^*) + d(x^*, T_{n_k+1}x_{n_k})] \\ &\leq \frac{\lambda}{2}[d(x_{n_k}, v) + d(x^*, x_{n_k+1})]. \end{aligned}$$

Let $k \to \infty$. Then $d(x^*, v) \leq \frac{\lambda}{2} d(x^*, v)$, and so $d((x^*, v) = 0$ since $\lambda < 1$, so $v = x^*$.

In any cases, we have that $v = x^*$, and so $x^* \in T_i x^*$. This means that x^* is a common fixed point of $\{T_i\}_{i \in \mathbb{N}}$.

If the condition (2.1) holds for all $i, j \in \mathbb{N}$ with $i \neq j$, $x, y \in K$, $u \in T_i x$ and $v \in T_j y$, then $\{T_i\}_{i \in \mathbb{N}}$ has a unique common fixed point. In fact, If x^* and y^* are all common fixed points of $\{T_i\}_{i \in \mathbb{N}}$ in K, then since $x^* \in T_1 x^*$ and $y^* \in T_2 y^*$, we have that

$$d(x^*, y^*) \leq \lambda u_{1,2}(x^*, y^*),$$

where, $u_{1,2}(x^*, y^*) \in \Big\{d(x^*, y^*), d(x^*, T_1x^*), d(y^*, T_2y^*),$

$$\frac{d(x^*, T_1x^*) + d(y^*, T_2y^*)}{2}, \frac{d(x^*, T_2y^*) + d(y^*, T_1x^*)}{2}\Big\}.$$

If $u_{1,2}(x^*, y^*) = d(x^*, y^*)$, then

$$d(x^*, y^*) \leq \lambda d(x^*, y^*),$$

and so $d(x^*, y^*) = 0$ since $\lambda < 1$. So $x^* = y^*$.

If $u_{1,2}(x^*, y^*) = d(x^*, T_1x^*)$, then

$$d(x^*, y^*) \leq \lambda d(x^*, T_1x^*) \leq \lambda d(x^*, x^*) = 0,$$

and so $x^* = y^*$.

If $u_{1,2}(x^*, y^*) = d(y^*, T_2y^*)$, then

$$d(x^*, y^*) \leq \lambda d(y^*, T_2y^*) \leq \lambda d(y^*, y^*) = 0,$$

and so $x^* = y^*$.

If $u_{1,2}(x^*, y^*) = \frac{d(x^*, T_1x^*) + d(y^*, T_2y^*)}{2}$, then

$$d(x^*, y^*) \leq \frac{\lambda}{2}[d(x^*, T_1x^*) + d(y^*, T_2y^*)] \leq \frac{\lambda}{2}[d(x^*, x^*) + d(y^*, y^*)] = 0,$$

and so $x^* = y^*$.

If $u_{1,2}(x^*, y^*) = \frac{d(x^*, T_2y^*) + d(y^*, T_1x^*)}{2}$, then

$$\begin{aligned} d(x^*, y^*) &\leq \frac{\lambda}{2}[d(x^*, T_2y^*) + d(y^*, T_1x^*)] \\ &\leq \frac{\lambda}{2}[d(x^*, y^*) + d(y^*, x^*)] \\ &= \lambda d(x^*, y^*), \end{aligned}$$

and $d(x^*, y^*) = 0$ since $\lambda < 1$. So $x^* = y^*$.

Hence we know that $x^* = y^*$ in any situation. So the common fixed points of $\{T_i\}_{i\in\mathbb{N}}$ are unique. □

REMARK 2.2. When $\{T_i\}_{i\in\mathbb{N}}$ are all single mappings, the sequence $\{x'_n\}$ can be constructed by the next way without using the condition (2.1): $x'_n = T_n x_{n-1}$ for all $n \geq 1$, see [9, 11]. Hence we can see that the our technique here is very different from that in [9, 11], and I think this is a new method.

From Theorem 2.1, we can easily obtain the following common fixed point theorem for a countable family of non-self single-valued mappings defined on a nonempty closed subset of a metrically convex space.

THEOREM 2.3. *Let K be a nonempty closed subset of a complete metrically convex space (X, d), $\{T_i : K \to X\}_{i\in\mathbb{N}}$ a countable family of non-self single-valued mappings such that for any $i, j \in \mathbb{N}$ with $i \neq j$, and any $x, y \in K$ satisfies*

$$d(T_i x, T_j y) \leq \lambda u_{i,j}(x, y), \tag{2.2}$$

where, $u_{i,j}(x,y) \in \{d(x,y), d(x,T_ix), d(y,T_jy), \frac{d(x,T_ix)+d(y,T_jy)}{2}, \frac{d(x,T_jy)+d(y,T_ix)}{2}\}$

and $\lambda \in (0, \frac{1}{2})$ is a constant number. Furthermore, if $T_i(x) \in K$ for all $i \in \mathbb{N}$ and $x \in \partial K$, then $\{T_i\}_{i\in\mathbb{N}}$ have a unique common fixed point in K.

We can obtain the following more general common fixed point theorem.

THEOREM 2.4. *Let K be a nonempty closed subset of a complete metrically convex space (X, d), $\{T_{i,j} : X \to X\}_{i,j\in\mathbb{N}}$ a family of non-self single-valued mappings, $\{m_{i,j}\}_{i,j\in\mathbb{N}}$ a family of positive integral numbers such that there exists a constant number $\lambda \in (0, \frac{1}{2})$ such that for each $x, y \in X$ and $i_1, i_2, j \in \mathbb{N}$ with $i_1 \neq i_2$,*

$$d(T_{i_1,j}^{m_{i_1,j}} x, T_{i_2,j}^{m_{i_2,j}} y) \leq \lambda\, u_{i_1,i_2,j}(x, y), \tag{2.3}$$

where, $u_{i_1,i_2,j}(x,y) \in \{d(x,y), d(x,T_{i_1,j}^{m_{i_1,j}}x), d(y,T_{i_2,j}^{m_{i_2,j}}y), \frac{d(x,T_{i_1,j}^{m_{i_1,j}}x)+d(y,T_{i_2,j}^{m_{i_2,j}}y)}{2}, \frac{d(x,T_{i_2,j}^{m_{i_2,j}}y)+d(y,T_{i_1,j}^{m_{i_1,j}}x)}{2}\}$. *Furthermore, suppose (a) for each $i, j \in \mathbb{N}$, $T_{i,j}^{m_{i,j}}(\partial K) \subset K$, (b) for each $i_1, i_2, \mu, \nu \in \mathbb{N}$ with $\mu \neq \nu$, $T_{i_1,\mu}T_{i_2,\nu} = T_{i_2,\nu}T_{i_1,\mu}$. Then $\{T_{i,j}\}_{i,j\in\mathbb{N}}$ has a unique common fixed point in K.*

Proof. Fix $j \in \mathbb{N}$, and let $S_{i,j} = T_{i,j}^{m_{i,j}}$, then $\{S_{i,j}\}_{i\in\mathbb{N}}$ satisfies all of the assumptions of Theorem 2.3. Hence $\{S_{i,j}\}_{i\in\mathbb{N}}$ has a unique common fixed point p_j in K. Now, we will prove that p_j is also a unique common fixed point of $\{T_{i,j}\}_{i\in\mathbb{N}}$. In fact, for any fixed $i \in \mathbb{N}$,

$$S_{i,j}(T_{i,j}(p_j)) = T_{i,j}^{m_{i,j}}(T_{i,j}(p_j)) = T_{i,j}(T_{i,j}^{m_{i,j}}(p_j)) = T_{i,j}(S_{i,j}(p_j)) = T_{i,j}(p_j).$$

This means that $T_{i,j}(p_j)$ is a fixed point of $S_{i,j}$. For any $k \in \mathbb{N}$ with $k \neq i$, we have

$$\begin{aligned} d(T_{i,j}(p_j), S_{k,j}(T_{i,j}(p_j))) &= d(S_{i,j}(T_{i,j}(p_j)),\\ S_{k,j}(T_{i,j}(p_j))) &\leq \lambda\, u_{i,k,j}(T_{i,j}(p_j), T_{i,j}(p_j)), \end{aligned}$$

where,

$$
\begin{aligned}
&u_{i,k,j}(T_{i,j}(p_j), T_{i,j}(p_j))\\
&\in \Big\{d(T_{i,j}(p_j), T_{i,j}(p_j)), d(T_{i,j}(p_j), S_{i,j}(T_{i,j}(p_j))), d(T_{i,j}(p_j), S_{k,j}(T_{i,j}(p_j))),\\
&\qquad \frac{d(T_{i,j}(p_j), S_{i,j}(T_{i,j}(p_j))) + d(T_{i,j}(p_j), S_{k,j}(T_{i,j}(p_j)))}{2}\Big\}\\
&= \Big\{0, d(T_{i,j}(p_j), S_{k,j}(T_{i,j}(p_j))), \frac{d(T_{i,j}(p_j), S_{k,j}(T_{i,j}(p_j)))}{2}\Big\}.
\end{aligned}
$$

If $u_{i,k,j}(T_{i,j}(p_j), T_{i,j}(p_j)) = 0$, then

$$d(T_{i,j}(p_j), S_{k,j}(T_{i,j}(p_j))) \leq \lambda\, 0 = 0,$$

and hence $d(T_{i,j}(p_j),\, S_{k,j}(T_{i,j}(p_j))) = 0$, i.e,,$T_{i,j}(p_j) = S_{k,j}(T_{i,j}(p_j))$.
If $u_{i,k,j}(T_{i,j}(p_j), T_{i,j}(p_j)) = d(T_{i,j}(p_j), S_{k,j}(T_{i,j}(p_j)))$, then

$$d(T_{i,j}(p_j), S_{k,j}(T_{i,j}(p_j))) \leq \lambda\, d(T_{i,j}(p_j), S_{k,j}(T_{i,j}(p_j))),$$

and so $T_{i,j}(p_j) = S_{k,j}(T_{i,j}(p_j))$.
If $u_{i,k,j}(T_{i,j}(p_j), T_{i,j}(p_j)) = \frac{d(T_{i,j}(p_j), S_{k,j}(T_{i,j}(p_j)))}{2}$, then

$$d(T_{i,j}(p_j), S_{k,j}(T_{i,j}(p_j))) \leq \frac{\lambda}{2} d(T_{i,j}(p_j), S_{k,j}(T_{i,j}(p_j))),$$

and so $T_{i,j}(p_j) = S_{k,j}(T_{i,j}(p_j))$. Hence in any situation, $T_{i,j}(p_j)$ is a fixed point of $S_{k,j}$ for each k with $k \neq i$. That is, $T_{i,j}(p_j)$ is a common fixed point of $\{S_{i,j}\}_{i\in\mathbb{N}}$. By the uniqueness of common fixed points of $\{S_{i,j}\}_{i\in\mathbb{N}}$, we have $T_{i,j}(p_j) = p_j$ for each $i \in \mathbb{N}$. Hence p_j is a common fixed point of $\{T_{i,j}\}_{i\in\mathbb{N}}$.

If u_j and v_j are all common fixed points of $\{T_{i,j}\}_{i\in\mathbb{N}}$, then they are also common fixed points of $\{S_{i,j}\}_{i\in\mathbb{N}}$. By the uniqueness of common fixed points of $\{S_{i,j}\}_{i\in\mathbb{N}}$, we obtain that $u_i = p_j = v_j$. This means that for each $j \in \mathbb{N}$, $\{T_{i,j}\}_{i\in\mathbb{N}}$ has a unique common fixed point p_j.

Finally, we will prove that $\{T_{i,j}\}_{i,j\in\mathbb{N}}$ has a unique common fixed point. First, we prove that for each $\mu, \nu \in \mathbb{N}$, $p_\mu = p_\nu$. In fact, for any $i_1, i_2, \mu, \nu \in \mathbb{N}$ with $\mu \neq \nu$, since $T_{i_1,\mu}(p_\mu) = p_\mu$ and $T_{i_2,\nu}(p_\nu) = p_\nu$,

$$T_{i_1,\mu}(T_{i_2,\nu}(p_\nu)) = T_{i_1,\mu}(p_\nu).$$

Hence by (b)

$$T_{i_2,\nu}(T_{i_1,\mu}(p_\nu)) = T_{i_1,\mu}(T_{i_2,\nu}(p_\nu)) = T_{i_1,\mu}(p_\nu).$$

This means that $T_{i_1,\mu}(p_\nu)$ is a fixed point of $T_{i_2,\nu}$ for each i_2, i.e., $T_{i_1,\mu}(p_\nu)$ is a common fixed point of $\{T_{i_2,\nu}\}_{i_2\in\mathbb{N}}$. Since $\{T_{i_2,\nu}\}_{i_2\in\mathbb{N}}$ has a unique common fixe point p_ν, we see that $T_{i_1,\mu}(p_\nu) = p_\nu$ for each i_1, and therefore p_ν is a common fixed point of $\{T_{i_1,\mu}\}_{i_1\in\mathbb{N}}$. But $\{T_{i_1,\mu}\}_{i_1\in\mathbb{N}}$ has an unique common fixed point p_μ, and so $p_\mu = p_\nu$. Let $p^* = p_j$. Then p^* is the common fixed point of $\{T_{i,j}\}_{i,j\in\mathbb{N}}$. The uniqueness of common fixed points of $\{T_{i,j}\}_{i,j\in\mathbb{N}}$ is obvious. □

From Theorem 2.1, we obtain next common fixed point theorem:

THEOREM 2.5. *Let K be a nonempty closed subset of a complete metrically convex space (X, d), $\{T_i : K \to k(X)\}_{i\in\mathbb{N}}$ a countable family of non-self set-valued mappings with nonempty values such that for any $i, j \in \mathbb{N}$ with $i \neq j$, and any $x, y \in K$ satisfies*

$$H(T_i x, T_i y) \le \lambda u_{i,j}(x, y), \tag{2.4}$$

where, $u_{i,j}(x, y) \in \left\{ d(x, y), d(x, T_i x), d(y, T_j y), \frac{d(x,T_i x)+d(y,T_j y)}{2}, \frac{d(x,T_j y)+d(y,T_i x)}{2} \right\}$ *and $\lambda \in (0, \frac{1}{2})$ is a constant number. Furthermore, if $T_i(x) \subset K$ for all $i \in \mathbb{N}$ and $x \in \partial K$, then $\{T_i\}_{i\in\mathbb{N}}$ have a common fixed point in K.*

Proof. For any $i, j \in \mathbb{N}$ and any $u \in T_i x$, by applying (iii) in Lemma 1.3, we can choose $v \in T_j y$ such that $d(u, v) \le H(T_i x, T_j y)$. Hence (2.1) in Theorem 2.1 holds by (2.4), So $\{T_i\}_{i\in\mathbb{N}}$ has a common fixed point in K by Theorem 2.1. □

References

[1] N. A. Assad, *Fixed point theorems for set-valued transformations on compact sets*, Boll. Un. Math. Ital. **7**(4) (1973), 1-7.

[2] N. A. Assad, *On fixed point theorem of Kannan in Banach spaces*, TamKang J. Math. **7** (1976), 91-94.

[3] N. A. Assad and W. A. Kirk, *Fixed point theorems for set-valued mappings of contractive type*, Pacific J. Math. **43** (1972), 553-562.

[4] S. K. Chatterjea, *Fixed point theorems*, C. R. Acad. Bulagarc Sci. **25** (1972), 727-730.

[5] O. Hadzic, *Common fixed point theorem for a family of mappings in convex metric spaces*, Univ. U. Novom Sadu, Zb. Rad. Prirod. Mat. Fak. Ser. Mat. **20**(1) (1990), 89-95.

[6] M. S. Khan, H. K. Pathak, and M. D. Khan, *Some fixed point theorems in metrically convex spaces*, Georgain J. Math. **7**(3) (2000), 523-530.

[7] I. S. Kim, *Common fixed point theorems in 2-metric spaces*, Master's Thesis, Korea Soongsil University, Seoul.

[8] A. Mujahid, B. E. Rhoades, and N. Talat, *Common fixed points of generalized contractive multivalued mappings in cone metric sapces*, Math. Commun. **14**(2) (2009), 365-378.

[9] Y. J. Piao, *Unique common fixed point theorems for a family of non-self maps in metrically convex spaces*, Math. Appl. **22**(4) (2009), 852-857.

[10] Y. J. Piao, *Unique common fixed points for a family of quasi-contractive type maps in metrically convex spaces*, Acta Mathematica Scientia. **30A(2)** (2010), 487-493.

[11] Y. J. Piao and D. Z. Piao, *A family of non-self maps satisfying ϕ_i-contractive condition and having unique common fixed point in metrically convex spaces*, Advances in Pure Mathematics. **2**(4) (2012), 280-284.

[12] S. L. Singh and B. Ram, *Common fixed points of commuting mappings in 2-metric spaces*, Math. Sem. Notes **10** (1982), 197-207.

[13] J. R. Wu and H. Y. Liu, *Common fixed point theorems for sequences of Φ-type contraction set-valued mappings*, Chin. Quart. J. Math. **24**(4) (2009), 504-510.

[14] X. Zhang, *Common fixed point theorem of Lipschitz type mappings on cone metric spaces*, Acta. Math. Sinica(Chinese Series). **53**(6) (2010), 1139-1148.

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